Solution Methods for Linear Differential Equations

Linear differential equations of order n have the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x). \tag{1}$$

The equation is homogeneous if $g(x) = 0$. Usually we'll only be able to solve the equation if it is first order or the left side has constant coefficients. It is convenient to think of the left-hand side as a linear differential operator, by which we mean an algebraic construction that uses derivatives of a function y to produce a new function $L[y]$ that uses the same independent variable. For example, if $L[y] = y'' + y$ and $y_p = e^{3x}$, then $y_p'' = 9e^{3x}$ and $L[y_p] = 10e^{3x}$. This confirms that $y_p = e^{3x}$ is a solution of the equation $y'' + y = 10e^{3x}$.

Stop for a moment and see if you can find a solution of $y'' + y = e^{3x}$. Use what we've already figured out to identify a function that gives the right answer for $L[y]$.

1. Generalized Exponential Functions

It is convenient to expand the class of exponential functions to a larger class of *generalized exponential functions*. Each of these functions is associated with a *characteristic value*, which can be either real or complex. In the table below, the notation $p_1(x)$ refers to any first degree polynomial. These are all the functions that can solve homogeneous linear equations.

Table 1: Generalized Exponential Functions

Functions	Characteristic Value(s)
1	0
e^{mx}	m
$\cos\beta x$	$\pm i\beta$
$\sin\beta x$	$\pm i\beta$
$e^{\alpha x}\cos\beta x$	$\alpha \pm i\beta$
$e^{\alpha x}\sin\beta x$	$\alpha \pm i\beta$
$p_1(x)$	$0,\ 0$
$p_1(x)e^{mx}$	$m,\ m$
$p_1(x)\cos\beta x$	$\pm i\beta,\ \pm i\beta$
$p_1(x)\sin\beta x$	$\pm i\beta,\ \pm i\beta$
$p_1(x)e^{\alpha x}\cos\beta x$	$\alpha \pm i\beta,\ \alpha \pm i\beta$
$p_1(x)e^{\alpha x}\sin\beta x$	$\alpha \pm i\beta,\ \alpha \pm i\beta$
$\vdots$	$\vdots$

Stop for a moment and check $L[e^{-x}\cos x] = 0$ where $L[y] = y'' + 2y' + 2y$. This confirms that $e^{-x}\cos x$ is a generalized exponential function.

2. Homogeneous Linear Equations

Homogeneous linear equations with constant coefficients have solutions that are generalized exponential functions. All we have to do to solve them is find the characteristic values for L and match them to the appropriate functions in the table. We find the characteristic values by finding the roots of the characteristic polynomial obtained by substituting $y = e^{mx}$ into $L[y]$.

Example 1: $y'' + 2y' + 2y = 0$
The characteristic equation is $m^2+2m+2 = 0$, or $m^2+2m+1 = -1$, where we have retained 1 on the left hand side to make a perfect square. Then $(m+1)^2 = -1$. Taking square roots, we have $m + 1 = \pm i$ or $m = -1 \pm i$. The associated functions are $e^{-x} \cos x$ and $e^{-x} \sin x$, so

$$y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x.$$

Example 2: $y'' + 2y' + y = 0$
The characteristic equation is $m^2 + 2m + 1 = 0$, or $(m + 1)^2 = 0$. Thus, $m = -1$ is a double root. The two solutions we need are e^{-x} and xe^{-x}, so

$$y = (c_1 + c_2 x)e^{-x}.$$

Stop for a moment and check $L[xe^{-x}] = 0$ *where* $L[y] = y'' + 2y' + y$.

To solve $L[y] = 0$ **where** L **has constant coefficients**:

1. Assume $y = e^{mx}$ and obtain the characteristic equation for the unknown m.
2. Find the roots of the characteristic polynomial, being careful to include pairs of complex roots and keeping track of the number of factors (multiplicity) of each root.
3. Write solutions for each root.
   - A real root m of multiplicity k gives solutions e^{mx}, xe^{mx}, …, $x^{k-1}e^{mx}$.
   - A complex pair $\alpha \pm i\beta$ of multiplicity k gives solutions $e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$, $xe^{\alpha x} \cos \beta x$, $xe^{\alpha x} \sin \beta x$, … $x^{k-1}e^{\alpha x} \cos \beta x$, $x^{k-1}e^{\alpha x} \sin \beta x$.

3. Nonhomogeneous Linear Equations

To solve $L[y] = g(x)$:

1. Find the general solution $y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ of the homogeneous equation $L[y] = 0$.
2. Find a particular solution y_p of the equation $L[y] = g(x)$.
3. The general solution of $L[y] = g(x)$ is $y = y_h + y_p$.

Two difficulties can arise in implementation of this method. First, it depends on our being able to solve the homogeneous equation. This is easy for first-order equations and all equations with constant coefficients, but it is often impossible in other cases. Second, finding a particular solution can be very difficult. We'll consider three ways of doing this.

4. Variation of Parameters for First-Order Equations

Variation of parameters can always be used for first-order linear equations. This is because the homogeneous equation is always separable so that a function y_1 and the homogeneous solution $y_h = c_1 y_1$ are known. While there are technical details, the method itself is straightforward.

To find y_p for $a_1(x)y' + a_0(x)y = g(x)$, given the general solution $y = c_1 y_1$ of $a_1(x)y' + a_0(x)y = 0$:

1. Construct the function $u'(x) = g(x)/y_1(x)$.
2. Integrate u' to get a function u.
3. Then $y_p = u(x)y_1(x)$.

Example 3: $y' + 2y = 4e^{2x}$
We have $y_1 = e^{-2x}$, $u' = 4e^{2x}/e^{-2x} = 4e^{4x}$, $u = e^{4x}$ and $y_p = e^{4x}e^{-2x} = e^{2x}$.

5. Variation of Parameters for Second-Order Equations

Variation of parameters can be used for second-order linear equations if we can first find the homogeneous solution $y_h = c_1 y_1 + c_2 y_2$. As with first-order equations, the method is straightforward, but the technical details can be problematic.

To find y_p for $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$, given the general solution $y = c_1 y_1 + c_2 y_2$ of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$:

1. Calculate the Wronskian $W(x) = y_1 y_2' - y_1' y_2$, calculate and simplify the function g/W, and construct the functions
$$u_2'(x) = y_1 \frac{g}{W}, \qquad u_1' = -y_2 \frac{g}{W}.$$
2. Integrate u_1' and u_2' to get functions u_1 and u_2.
3. Then $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$. This formula should always be simplified if possible.

Example 4: $y'' - 3y' + 2y = e^{2x}$
The homogeneous problem yields $y_1 = e^x$ and $y_2 = e^{2x}$. Then $W = e^{3x}$ and $g/W = e^{-x}$. So $u_2' = 1$ and $u_1' = -e^x$. Integrating gives us $u_1 = -e^x$ and $u_2 = x$. The final result is $y_p = -e^{2x} + xe^{2x}$. This is a correct particular solution, but it is not the simplest one. The term $-e^{2x}$ is not needed because it is part of the homogeneous solution $c_1 e^x + c_2 xe^{-x}$. So we could just use $y_p = xe^{2x}$.

Stop for a moment and check that $y_p = xe^{2x}$ is in fact a particular solution. Also check the result for the Wronskian.

6. Undetermined Coefficients for Equations with Constant Coefficients and g a Generalized Exponential Function

A general rule of thumb in mathematics is that the more a method makes use of the special structure of a problem, the easier it is likely to be to use the method. This is the principle of the method of undetermined coefficients. Subject to an important correction, the idea is that if you want a particular type of generalized exponential function to come out of a linear differential operator with constant coefficients, you have to put in the same type of function.

Example 5: $L[y] = y'' + 4y = 8x^3$
Note that (after simplifying) $L[x^3] = 4x^3+6$, $L[x^3+x^2] = 4x^3+4x^2+6x+2$, and $L[2x^3-3x] = 8x^3$. So we got lucky: the particular solution is $y_p = 2x^3 - 3x$. Unfortunately, we can't solve problems by guessing. What we can do is identify the structure of the solution and put it to use. In each of the three examples, the input function was a cubic polynomial and the output function was also a cubic polynomial. If we were solving this problem without pre-knowledge of the answer, all we would know without doing any calculations is that $L[Ax^3 + Bx^2 + Cx + D]$ is a 4-parameter family of cubic polynomials. Then all we have to do is find the coefficients A, B, C, and D. We have

$$y = Ax^3 + Bx^2 + Cx + D, \quad y' = 3Ax^2 + 2Bx + C, \quad y'' = 6Ax + 2B,$$

and so

$$L[Ax^3 + Bx^2 + Cx + D] = 4Ax^3 + 4Bx^2 + (6A + 4C)x + (8B + D).$$

If we choose just the right values of the four coefficients, we can get $L[Ax^3+Bx^2+Cx+D] = 8x^3$. This requires

$$4A = 8, \qquad 4B = 0, \qquad 6A + 4C = 0, \qquad 8B + D = 0.$$

Thus $A = 2$, $B = 0$, $C = -3$, and $D = 0$, yielding the solution $y = 2x^3 - 3x$.

If you think Example 5 was hard work, then stop for a moment and try solving the problem with variation of parameters. Be prepared to integrate by parts six times only to see most of the results cancel when you combine u_1y_1 and u_2y_2!

The method of undetermined coefficients requires a bit more nuance. Look back at Example 4. We used variation of parameters to find the particular solution $y_p = xe^{2x}$ for $y''-3y'+2y = e^{2x}$. This doesn't look right. Based on our earlier work, we should expect to put in a constant times e^{2x} and get back a different constant times e^{2x}. But that isn't what happens: instead e^{2x} solves $y''-3y'+2y = 0$. The problem is that there is a conflict between the characteristic values of y_h (1 and 2) and the characteristic value of g (2). When the characteristic value of g is a root of multiplicity k of the characteristic equation, then L reduces the degree of the input polynomial by k. Here, 2 is a root of multiplicity 1. If we put in a constant times e^{2x}, we get nothing. If we put in Axe^{2x}, the result should be a constant times e^{2x}, which is what we want. The structure of the problem tells us that we should use $y_p = Axe^{2x}$ for this L, not $y_p = Ae^{2x}$.

To find y_p for $L[y] = g(x)$, given an operator L with constant coefficients, along with a generalized exponential function g:

1. Start by constructing the most general function that has the same characteristic value as g and the same degree of polynomial factor.

2. If the characteristic value of g matches a characteristic value of L having multiplicity k, then the initial form for y_p must be multiplied by x^k.

3. Given your form for y_p, compute $L[y_p]$ and set it equal to g. If your form was correct, you will get just the right number of equations and be able to solve them uniquely for the unknown coefficients.

Example 6: $L[y] = y'' + 2y' + y = 6xe^{-x}$
The function g has characteristic value -1 and degree 1, so our first try is $y_p = (Ax+B)e^{-x}$, which is the most general form that includes a product of a first degree polynomial and e^{-x}. However, the characteristic equation for this L is $(m+1)^2 = 0$, making -1 a root of multiplicity 2 and putting e^{-x} and xe^{-x} into the homogeneous solution. So we have to multiply our initial form by x^2. The correct form for y_p is $y_p = (Ax^3 + Bx^2)e^{-x}$. Applying the method yields the particular solution $y_p = x^3e^{-x}$.

Stop for a moment and make sure you know how to complete the calculation of y_p. Compute $L[(Ax^3 + Bx^2)e^{-x}]$, set the result equal to g, and solve for A and B to get the answer $y_p = x^3e^{-x}$.

Example 7: $L[y] = y'' + 2y' + y = 25\cos 2x$
The function g has characteristic values $\pm 2i$ and the polynomial degree is 0, so we start with $y_p = A\cos 2x + B\sin 2x$, which is the most general form for this pair of characteristic values and degree 0. Fortunately $\pm 2i$ is not a characteristic value pair for L, so we have the right form. After simplifying, we get

$$L[A\cos 2x + B\sin 2x] = (4B - 3A)\cos 2x + (-4A - 3B)\sin 2x.$$

The output needs to be $\cos x$, so A and B have to satisfy the equations

$$4B - 3A = 25, \qquad -4A - 3B = 0.$$

The simplest way to solve these equations is to multiply the first by 3 and the second by 4:

$$12B - 9A = 75, \qquad -16A - 12B = 0.$$

After this multiplication, the coefficients of B in the two equations are 12 and -12. Adding the two equations makes B disappear:

$$0B - 25A = 75.$$

So $A = -3$. Then $3B = -4A = 12$, so $B = 4$. The particular solution is

$$y_p = -3\cos 2x + 4A\sin 2x.$$